



Tennessee Higher Education Commission

Supplementary Task: Examining the End Behavior of Rational Functions

Students often do not understand the connection between the end behavior of polynomials and the end behavior of rational functions. Often, discussions of rational functions in basic college algebra courses center on graphing the function, paying particular attention to finding the zeroes and asymptotes of the function. In many cases, students are given a set of “rules” for finding the horizontal asymptote or slant asymptote (if either one exists) with no context of why these “rules” are valid. This task is designed to support students develop a deeper understanding of the graph of a rational function by examining the end behavior and should be used near the beginning of the study of rational functions.

These examples support the objective:

Use functions to model and solve problems using tables, graphs, and algebraic properties. Interpret constants, coefficients, and bases in the context of the problem.

in the Core to College Master Syllabus for Core-Aligned College Algebra.

Example: Examining the End Behavior of Rational Functions

For each of the following rational functions:

- Rewrite the rational function as the sum of a polynomial and a rational function whose numerator has a smaller degree than its denominator. (Remember, a polynomial can be a constant.)
- Graph the original rational function using your graphing calculator.
- Graph the polynomial portion of your rewritten form of the original rational function.
- Discuss any relationships you see between the end behavior of the polynomial and the end behavior of the rational function.
- Discuss how you can predict the end behavior of the graph of the rational function without rewriting the function.

$$g(x) = \frac{5x^2 - 8}{2x^2 + 1}$$

$$G(x) = \frac{12x - 5}{3x + 2}$$

$$h(x) = \frac{3x^3 - 24}{x^2 + 1}$$

$$H(x) = \frac{x^3 - 2}{x + 1}$$

Prior Knowledge Needed:

Students need a basic understanding of the relationship between power functions and the end behavior of polynomials.

Students need to know how to divide one polynomial by another polynomial to get a quotient consisting of a polynomial portion and a portion that is a rational function with the degree of the numerator less than the degree of the denominator.

Students need to have a basic understanding of using a graphing calculator to graph polynomials and rational functions. Students may need to change the window settings in order to make the connections between the end behaviors of the graphs.

Prerequisite Common Core State Standards for Mathematical Content that support this example

Interpret the structure of expressions

Write expressions in equivalent forms to solve problems

Rewrite rational expressions

Interpret functions that arise in applications in terms of the context

Analyze functions using different representations

*The complete Common Core State Standards for High School Mathematics can be found in <http://www.corestandards.org/Math/>.

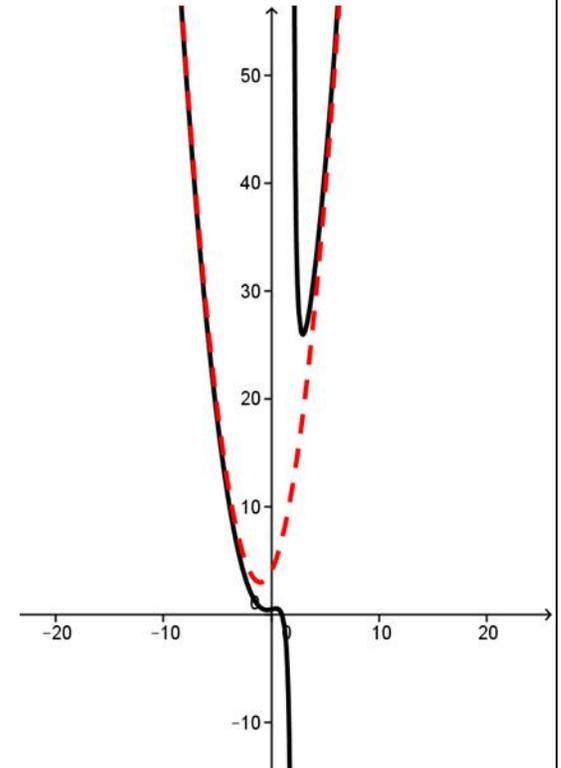
Solutions

Function $f(x)$:

(i): $f(x) = \frac{x^3 - 1}{x - 2} = x^2 + 2x + 4 + \frac{7}{x - 2}$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the quadratic function closely follows the end behavior of the graph of the rational function.

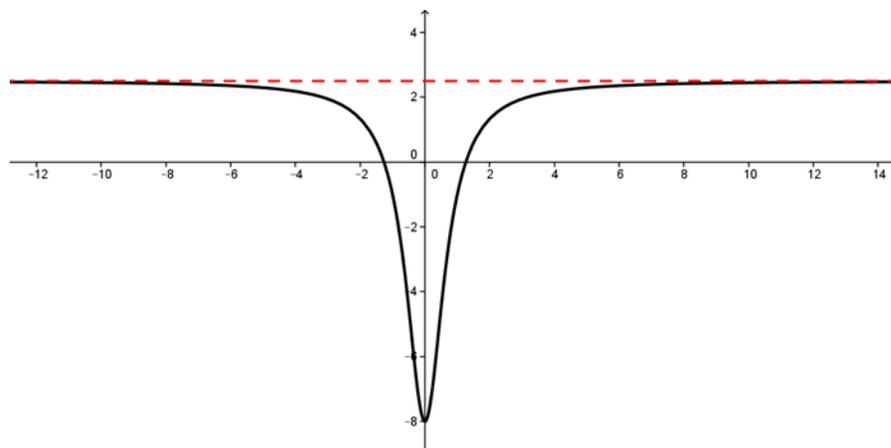


Function $g(x)$:

(i): $g(x) = \frac{5x^2 - 8}{2x^2 + 1} = \frac{5}{2} - \frac{\frac{21}{2}}{2x^2 + 1}$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the constant function closely follows the end behavior of the graph of the rational function.

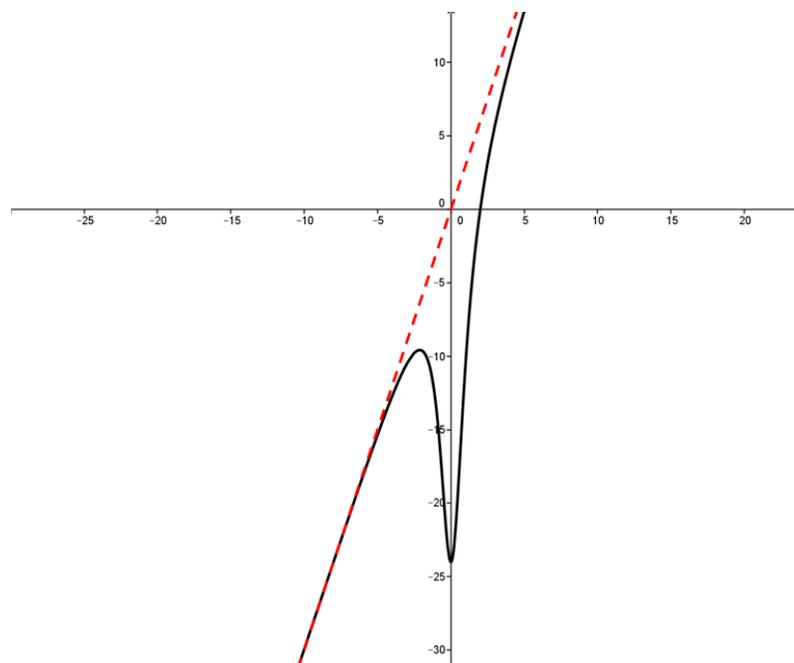


Function $h(x)$:

(i): $h(x) = \frac{3x^3 - 24}{x^2 + 1} = 3x - \frac{3x + 24}{x^2 + 1}$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the linear function closely follows the end behavior of the graph of the rational function.

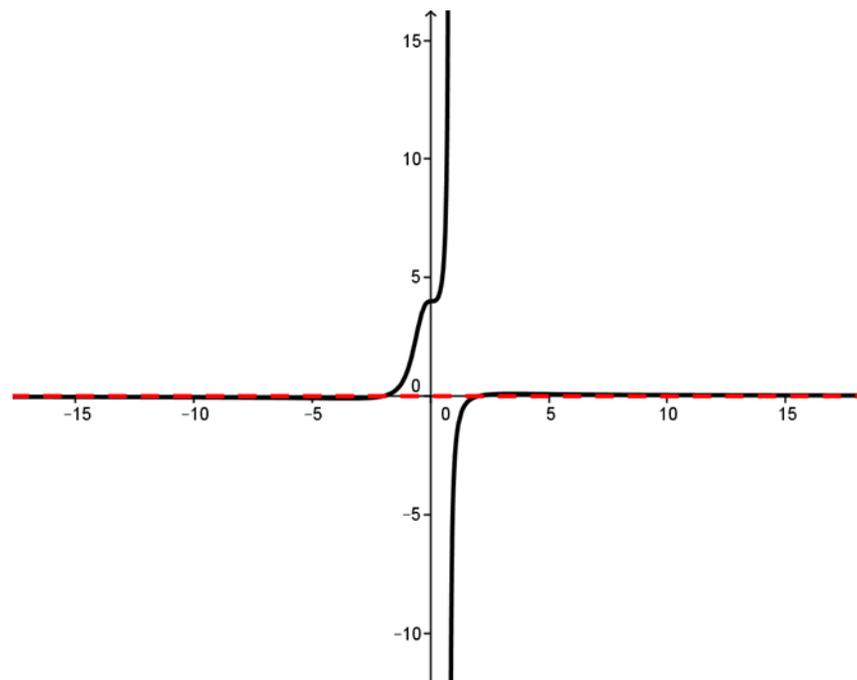


Function F(x):

(i):
$$F(x) = \frac{x^2 - 4}{2x^3 - 1} = 0 + \frac{x^2 - 4}{2x^3 - 1}$$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the zero function closely follows the end behavior of the graph of the rational function.

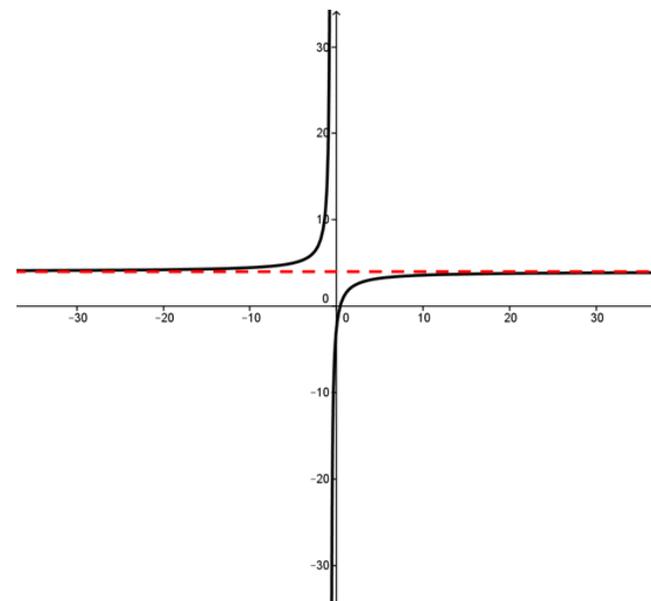


Function G(x):

(i):
$$G(x) = \frac{12x - 5}{3x + 2} = 4 - \frac{13}{3x + 2}$$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the constant function closely follows the end behavior of the graph of the rational function.

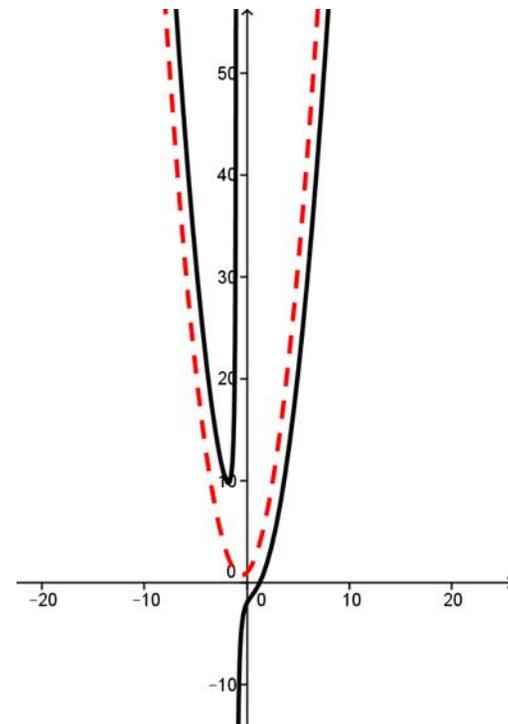


Function $H(x)$:

(i): $H(x) = \frac{x^3 - 2}{x + 1} = x^2 + x + 1 - \frac{1}{x - 1}$

(ii) and (iii): The function (in black) and the polynomial part (in red) are graphed to the right.

(iv): The graph of the quadratic function closely follows the end behavior of the graph of the rational function.



Tasks for Student Work

Predict the end behavior of each of these functions and justify your answer.

$$f(x) = \frac{8x + 3}{2x - 1}$$

$$g(x) = \frac{x}{x^2 - 9}$$

$$h(x) = \frac{x^2 + x + 3}{x + 1}$$

$$k(x) = \frac{5 - x}{x + 2}$$

Examining the End Behavior of Rational Functions Classroom Task

For each of the following rational functions:

- i. Rewrite the rational function as the sum of a polynomial and a rational function whose numerator has a smaller degree than its denominator. (Remember, a polynomial can be a constant.)
- ii. Graph the original rational function using your graphing calculator.
- iii. Graph the polynomial portion of your rewritten form of the original rational function.
- iv. Discuss any relationships you see between the end behavior of the polynomial and the end behavior of the rational function.
- v. Discuss how you can predict the end behavior of the graph of the rational function without rewriting the function.

$$f(x) = \frac{x^3 - 1}{x - 2}$$

$$F(x) = \frac{x^2 - 4}{2x^3 - 1}$$

$$g(x) = \frac{5x^2 - 8}{2x^2 + 1}$$

$$G(x) = \frac{12x - 5}{3x + 2}$$

$$h(x) = \frac{3x^3 - 24}{x^2 + 1}$$

$$H(x) = \frac{x^3 - 2}{x + 1}$$

Examining the End Behavior of Rational Functions Student Work

Predict the end behavior of each of these functions and justify your answer.

$$f(x) = \frac{8x+3}{2x-1}$$

$$g(x) = \frac{x}{x^2-9}$$

$$h(x) = \frac{x^2+x+3}{x+1}$$

$$k(x) = \frac{5-x}{x+2}$$